TRINITY COLLEGE

IN NOMINE DOMINI

**Semester Two Examination, 2020**

**Question/Answer booklet**

# MATHEMATICS METHODS UNITS 3&4

## Section One: Calculator-free

# SOLUTIONS

| | | |
|---|---|---|
| WA student number: | In figures | |
| | In words | ______________________ |
| | Your name | ______________________ |

## Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

## Materials required/recommended for this section

***To be provided by the supervisor***

This Question/Answer booklet
Formula sheet

***To be provided by the candidate***

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 8 | 8 | 50 | 52 | 35 |
| Section Two: Calculator-assumed | 13 | 13 | 100 | 98 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

**Section One: Calculator-free** **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

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**Question 1** **(7 marks)**

(a) Determine an expression for $f'(x)$ when

(i) $f(x) = \ln(1 - \cos 3x)$. (2 marks)

| Solution |
|---|
| $f'(x) = \dfrac{3\sin 3x}{1 - \cos 3x}$ |
| **Specific behaviours** |
| ✓ numerator<br>✓ denominator |

(ii) $f(x) = e^{5x}(5 - 2x)^3$. (3 marks)

| Solution |
|---|
| $f'(x) = 5e^{5x}(5-2x)^3 + e^{5x} \cdot 3(-2)(5-2x)^2$<br>$= 5e^{5x}(5-2x)^3 - 6e^{5x}(5-2x)^2$<br><br>*N.B. Simplifies to* $(19 - 10x)(5-2x)^2 e^{5x}$ |
| **Specific behaviours** |
| ✓ derivative of $e^{5x}$<br>✓ derivative of $(5-2x)^3$<br>✓ correct expression using product rule |

(b) For the positive number $x$, let $A(x) = \int_0^x (8 - 2^{t^2})\, dt$.

Determine the value(s) of $x$ for which $\dfrac{dA}{dx} = 0$. (2 marks)

| Solution |
|---|
| $\dfrac{dA}{dx} = \dfrac{d}{dx}\int_0^x (8 - 2^{t^2})\, dt$<br>$= 8 - 2^{x^2}$<br><br>$\therefore 2^{x^2} = 8 = 2^3 \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}$ |
| **Specific behaviours** |
| ✓ expression for $A'(x)$<br>✓ correct value of $x$ |

**Question 2** **(5 marks)**

The rate of change of pressure in an air tank is given by $P'(t) = -3e^{-0.05t}$, where $t$ is the time in minutes since it began emptying from an initial pressure of $70$ psi.

(a) Determine an expression for the pressure $P$ in the tank at any time $t,\ t \geq 0$. (2 marks)

| Solution |
|---|
| $P(t) = \frac{-3}{-0.05}e^{-0.05t} + c$ <br> $= 60e^{-0.05t} + c$ <br><br> $(0, 70) \Rightarrow 70 = 60e^{0} + c$ <br> $c = 10$ <br><br> $P(t) = 10 + 60e^{-0.05t}$ |
| **Specific behaviours** |
| ✓ correctly integrates $P'(t)$ <br> ✓ correct expression for $P(t)$ |

(b) Determine

(i) the time taken for the pressure in the tank to fall to $40$ psi. (2 marks)

| Solution |
|---|
| $10 + 60e^{-0.05t} = 40$ <br> $e^{-0.05t} = 0.5$ <br><br> $-0.05t = \ln 0.5$ <br> $t = -20\ln 0.5 \quad (= 20\ln 2)$ |
| **Specific behaviours** |
| ✓ simplifies equation to $e^{-0.05t} = k$ <br> ✓ correct time |

(ii) the minimum pressure in the tank for $t \geq 0$. (1 mark)

| Solution |
|---|
| $t \to \infty, P \to 10$ psi |
| **Specific behaviours** |
| ✓ correct pressure |

**Question 3** **(6 marks)**

The continuous random variable $X$ takes values in the interval $1$ to $5$ and has cumulative distribution function $F(x)$ where

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 1 \\ \dfrac{x-1}{4} & 1 \le x \le 5 \\ 1 & x > 5. \end{cases}$$

(a) Determine

(i) $P(X \le 3.5)$. (1 mark)

| Solution |
|---|
| $P(X \le 3.5) = \dfrac{3.5-1}{4} = \dfrac{2.5}{4} = \dfrac{5}{8} = 0.625$ |
| **Specific behaviours** |
| ✓ correct probability as fraction or decimal |

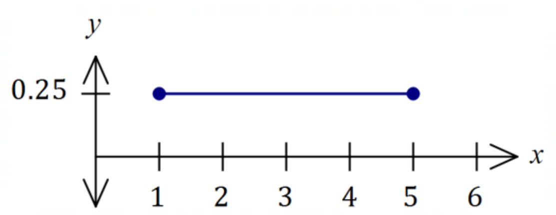
(ii) the value of $k$, if $P(X > k) = 0.85$. (2 marks)

| Solution |
|---|
| $P(X \le k) = 1 - P(X > k) = 1 - 0.85 = 0.15$ <br> $\dfrac{k-1}{4} = 0.15 \Rightarrow k = 1.6$ |
| **Specific behaviours** |
| ✓ indicates $P(X \le k)$ <br> ✓ correct value |

(b) Determine $f(x)$, the probability density function of $X$, and sketch the graph of $y = f(x)$. (3 marks)

**Solution**

$$f(x) = F'(x) = \begin{cases} \frac{1}{4} & 1 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$




**Specific behaviours**

✓ $f(x)$ (no penalty for just $f(x) = \frac{1}{4}$ if sketch correct)
✓ draws $y = 0.25$ between endpoints
✓ scaled and labelled axes

**Question 4** **(7 marks)**

The function $f$ is defined by $f(x) = \dfrac{x^2 - 5}{3 - x},\ x \neq 3$.

The second derivative of $f$ is $f''(x) = 8(3 - x)^{-3}$.

Determine the coordinates and nature of all stationary points of the graph of $y = f(x)$.

| Solution |
| --- |
| $f'(x) = \dfrac{2x(3-x) - (-1)(x^2-5)}{(3-x)^2}$ <br> $f'(x) = 0 \Rightarrow 6x - 2x^2 + x^2 - 5 = 0$ <br> $(x^2 - 6x + 5) = 0$ <br> $(x-1)(x-5) = 0$ <br> $x = 1, 5$ <br> $f''(1) = \dfrac{8}{8} > 0 \Rightarrow \text{Min},\quad f''(5) = \dfrac{8}{-8} < 0 \Rightarrow \text{Max}$ <br> $f(1) = -\dfrac{4}{2} = -2,\quad f(5) = \dfrac{20}{-2} = -10$ <br> $f(x)$ has a minimum at $(1, -2)$ and a maximum at $(5, -10)$. |
| **Specific behaviours** |
| ✓ indicates correct use of quotient rule <br> ✓ correct $f'(x)$ <br> ✓ equates numerator to zero <br> ✓ determines $x$-coordinates of stationary points <br> ✓ indicates correct use of second derivative for nature <br> ✓ correct minimum <br> ✓ correct maximum |

**Question 5** **(7 marks)**

(a) Simplify $\log 8 + 2\log 5 - \log 2$. (2 marks)

| Solution |
|---|
| $\log 8 + \log 5^2 - \log 2 = \log(8 \times 25 \div 2)$ <br> $= \log 10^2$ <br> $= 2$ |
| **Specific behaviours** |
| ✓ expresses as single log <br> ✓ simplifies to number |

(b) Given that $\log_a x = 1.4$, determine the value of $\log_a x\sqrt{x}$. (2 marks)

| Solution |
|---|
| $\log_a x\sqrt{x} = \log_a x + \log_a \sqrt{x}$ <br> $= \log_a x + 0.5\log_a x$ <br> $= 1.5\log_a x$ <br> $= 1.5 \times 1.4$ <br> $= 2.1$ |
| **Specific behaviours** |
| ✓ obtains multiple of $\log_a x$ <br> ✓ correct value |

(c) Determine the solution to the equation $5^{2x} = 2^{3-x}$ in the form $x = \dfrac{\log a}{\log b}$. (3 marks)

| Solution |
|---|
| $2x\log 5 = (3 - x)\log 2$ <br> $2x\log 5 = 3\log 2 - x\log 2$ <br> $2x\log 5 + x\log 2 = 3\log 2$ <br> $x(2\log 5 + \log 2) = 3\log 2$ <br> $x = \dfrac{\log 2^3}{\log(5^2 \times 2)}$ <br> $x = \dfrac{\log 8}{\log 50}$ |
| **Specific behaviours** |
| ✓ writes as log equation <br> ✓ factors out $x$ <br> ✓ solves and simplifies into required form |

**Question 6** **(7 marks)**

The discrete random variable $X$ is defined by

$$P(X=x)=\begin{cases}\dfrac{x+k}{3x+2} & x=0,1\\ 0 & \text{elsewhere}\end{cases}$$

(a) Determine the value of the constant $k$. (2 marks)

| Solution |
|---|
| $\frac{k}{2}+\frac{1+k}{5}=1$<br>$5k+2+2k=10$<br>$k=\frac{8}{7}$ |
| **Specific behaviours** |
| ✓ forms equation using $x=0$ and $x=1$<br>✓ correct value |

(b) Determine

(i) $E(X)$. (1 mark)

| Solution |
|---|
| $E(X)=p=P(X=1)=1-\frac{4}{7}=\frac{3}{7}$ |
| **Specific behaviours** |
| ✓ indicates $E(X)$ |

(ii) $E(3X-1)$. (2 marks)

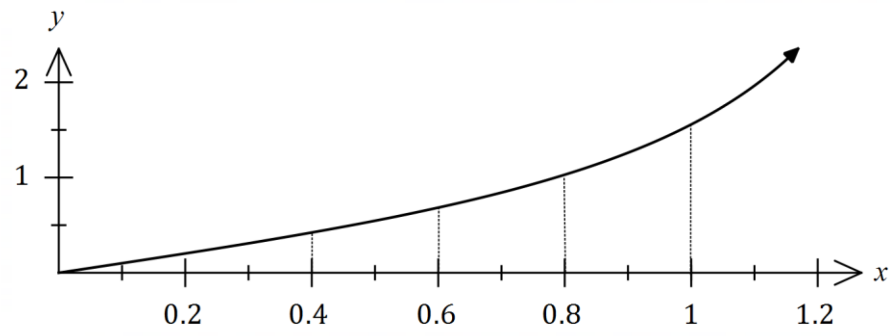
| Solution |
|---|
| $E(3X-1)=3\times\frac{3}{7}-1=\frac{2}{7}$ |
| **Specific behaviours** |
| ✓ correct value |

(c) Determine $\text{Var}(3X-1)$. (2 marks)

| Solution |
|---|
| $\text{Var}(X)=p(1-p)=\frac{3}{7}\times\frac{4}{7}=\frac{12}{49}$<br>$\text{Var}(3X-1)=3^2\times\frac{12}{49}=\frac{108}{49}$ |
| **Specific behaviours** |
| ✓ indicates $\text{Var}(X)$<br>✓ correct value |

**Question 7** **(7 marks)**

The graph and a table of values for $y = f(x)$ is shown below, where $f(x) = \tan x$.




| $x$ | $y$ |
|---|---|
| 0.2 | 0.2 |
| 0.4 | 0.42 |
| 0.6 | 0.68 |
| 0.8 | 1.03 |
| 1 | 1.56 |
| 1.2 | 2.57 |

Let $I = \int_{0.4}^{1} \tan x \, dx$.

(a) By using the information shown and considering sums of the form $\sum_i f(x_i)\delta x_i$, explain why $I > 0.426$. (3 marks)

| Solution |
|---|
| With $\delta x = 0.2, x_1 = 0.4, x_2 = 0.6$ and $x_3 = 0.8$ then<br>$\Sigma_i f(x_i)\delta x_i = 0.2(0.42 + 0.68 + 1.03)$<br>$= 0.2(2.13)$<br>$= 0.426$<br>Hence $I$ must exceed this value as it is the area of inscribed rectangles that underestimate the area under the curve. |
| **Specific behaviours** |
| ✓ indicates $x$-ordinates for inscribed rectangles<br>✓ shows sum of $f(x_i)\delta x_i$<br>✓ explains inequality |

(b) In a similar manner to (a), determine the best estimate for the value of the constant $U$, where $I < U$. (2 marks)

| Solution |
|---|
| With $\delta x = 0.2, x_1 = 0.6, x_2 = 0.8$ and $x_3 = 1$ then<br>$U = \Sigma_i f(x_i)\delta x_i = 0.2(0.68 + 1.03 + 1.56)$<br>$= 0.2(3.27)$<br>$= 0.654$ |
| **Specific behaviours** |
| ✓ indicates $x$-ordinates for circumscribed rectangles<br>✓ value of $U$ |

(c) Use your previous answers to determine a numerical estimate for $I$ and explain whether your estimate is smaller or larger than the exact value of $I$. (2 marks)

| Solution |
|---|
| $I = \dfrac{0.426 + 0.654}{2} = 0.54$<br>This value slightly overestimates the exact value of $I$ as the curve is concave upwards. |
| **Specific behaviours** |
| ✓ correctly averages values<br>✓ states overestimate with reason |

**Question 8** **(6 marks)**

The acceleration at time $t$ seconds of a small body travelling in a straight line is given by

$$a(t) = \frac{-3}{\sqrt{2t+3}} \text{ cm/s}^2, \qquad t \geq 0.$$

When $t = 3$ the body was at the origin and $8$ seconds later its displacement was $30$ cm.

Determine the velocity of the body when $t = 6.5$.

| Solution |
|---|
| $v(t) = \int -3(2t+3)^{-\frac{1}{2}}\,dt$ <br> $= \frac{-3}{\frac{1}{2}\times 2}(2t+3)^{\frac{1}{2}} + c$ <br> $= -3(2t+3)^{\frac{1}{2}} + c$ <br><br> $\Delta x = \int_3^{3+8} -3(2t+3)^{\frac{1}{2}} + c\,dt$ <br> $= \left[\frac{-3}{\frac{3}{2}\times 2}(2t+3)^{\frac{3}{2}} + ct\right]_3^{11}$ <br> $= \left[-(2t+3)^{\frac{3}{2}} + ct\right]_3^{11}$ <br> $= [-125 + 11c] - [-27 + 3c]$ <br> $= 8c - 98$ <br> But $\Delta x = 30$ <br> $8c - 98 = 30$ <br> $8c = 128$ <br> $c = 16$ <br><br> $v(6.5) = -3(2(6.5)+3)^{\frac{1}{2}} + 16$ <br> $= -12 + 16$ <br> $= 4 \text{ cm/s}$ |
| **Specific behaviours** |
| ✓ antiderivative of $a(t)$ <br> ✓ integral for change in displacement $\Delta x$ <br> ✓ antiderivative of $v(t)$ <br> ✓ simplifies equation for $c$ <br> ✓ uses given $\Delta x$ to determine value of $c$ <br> ✓ correct velocity |

**End of questions**

Supplementary page

Question number: _________